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LETTER TO THE EDITOR

Diagonalisation of corner transfer matrix by orthogonal polynomials

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Abstract. We discuss the generator of Baxter's corner transfer matrix for a critical Ising model of finite size. We perform its diagonalisation in terms of special orthogonal polynomials and give an asymptotic expression of the eigenvalues for large size, which agrees with the conformal predictions.

In the study of exactly solvable models in two-dimensional statistical mechanics the corner transfer matrix (CTM) turns out to be a powerful tool [1, 2]. This is mainly due to its surprisingly simple eigenvalue spectrum. Denoting the CTM by $\mathscr{A}$ it was found that, for an infinite system, the low-lying eigenvalues of $\ln \mathscr{A}$ are equidistant at all temperatures. At the critical point the level splitting vanishes and then the spectrum of a finite system is of interest. This has been discussed using conformal invariance [3]. For an isotropic Ising model defined on an area with the shape of an annular sector, the level splitting was found to be

$$\hat{\varepsilon} = \theta \frac{\pi}{2} \frac{1}{\ln R/a} \tag{1}$$

where R and a are the outer and inner radii of the annular sector and θ its opening angle. In this continuum limit, $\ln \mathscr{A}$ is also proportional to the element L_0 of the Virasoro algebra [4, 5].

The result (1) has already been tested numerically on a lattice, where R/a is to be replaced by N, the number of spins along an edge [6]. In the present letter we treat the problem analytically. To this end we study the XY-spin chain:

$$\mathscr{H} = \tfrac{1}{2} \sum_{n=1}^{N-1} n(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) \tag{2}$$

where the $\sigma_n^{x,y}$ are Pauli matrices. This operator is related to the CTM of two interpenetrating Ising lattices shown in figure 1. If the systems are strongly anisotropic ($K_2 \ll 1, K_1 \gg 1$) and one uses dual variables (i.e. the arrow or vertex representation [1]), the relation is $\mathscr{A} = \exp(-2K_1^* \mathscr{H})$ with $\tanh K_1^* = \exp(-2K_1)$. Baxter has shown that the relation $\ln \mathscr{A} \sim \mathscr{H}$ even holds for arbitrary anisotropy [1, 2], and therefore $\mathscr{H}$ is the typical operator to study. Note that we have chosen fixed (resp free) boundary conditions for the two sublattices and a particular shape of the outer boundary.

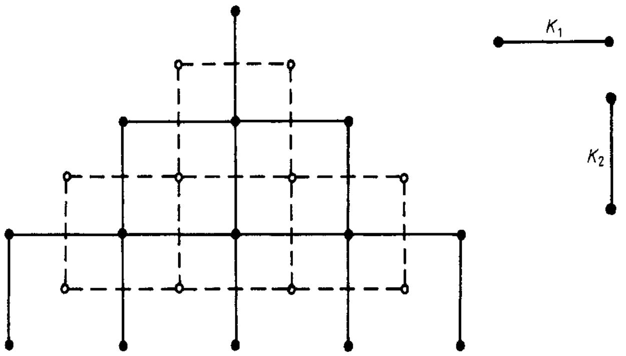

Figure 1. Geometry of the two interpenetrating Ising lattices which lead to the corner transfer matrix considered in the text. The system shown corresponds to $N=6$. The couplings are also indicated.

In terms of fermions, $\mathscr{H}$ becomes a hopping model with linearly increasing hopping rates

$$\mathscr{H}=\sum_{n=1}^{N-1} n(c_n^+c_{n+1}+c_{n+1}^+c_n). \tag{3}$$

A related operator with constant rates but on-site terms $nc_n^+c_n$ has been studied by Smith [7]. $\mathscr{H}$ can be diagonalised by forming linear combinations $\Sigma_n\psi_nc_n$. The coefficients ψ_n satisfy the recursion relation

$$(n-1)\psi_{n-1}(\lambda)+n\psi_{n+1}(\lambda)=\lambda\psi_n(\lambda) \qquad n=1,2,\ldots \tag{4}$$

together with $\psi_{N+1}(\lambda)=0$, which selects the single-fermion eigenvalues $\lambda=\lambda_\nu$ for a finite chain of N sites. To obtain the ψ_n explicitly, we introduce the generating function:

$$G(t,\lambda)=\sum_{n=0}^{\infty} t^n\psi_{n+1}(\lambda) \tag{5}$$

which in view of (4) obeys a simple first-order differential equation in t. Integrating this equation with the assumption $\psi_1(\lambda)=1$ gives

$$G(t,\lambda)=\frac{1}{(1+t^2)^{1/2}}\exp(\lambda\tan^{-1}t). \tag{6}$$

Then $\psi_n(\lambda)$ follows from Cauchy's theorem

$$\psi_n(\lambda)=\frac{1}{2\mathrm{i}\pi}\oint t^{-n}G(t,\lambda)\,\mathrm{d}t. \tag{7}$$

One now inserts (6), substitutes $t=\tan v$ and deforms the contour into a rectangle with vertical lines at $v=\pm\pi/2$. To make contact with standard notations [8] one defines $\psi_{n+1}(\lambda)=M_n(\lambda)/n!$ Then $M_n(\lambda)$ is given by

$$M_n(\lambda)=\frac{n!}{\pi}\cosh(\pi\lambda/2)\exp(-\mathrm{i}n\pi/2)\int_{-\infty}^{\infty}\frac{\tanh^n(y)}{\cosh y}\exp(\mathrm{i}y\lambda)\,\mathrm{d}y. \tag{8}$$

The $M_n(\lambda)$ are called Meixner polynomials of the second kind [8, 9]. The integral representation (8) differs from a related one given by Hardy [10]. The first $M_n(\lambda)$ are

$$\begin{aligned}&M_0(\lambda)=1 \qquad M_1(\lambda)=\lambda \qquad M_2(\lambda)=\lambda^2-1\\ &M_3(\lambda)=\lambda^3-5\lambda \qquad M_4(\lambda)=\lambda^4-14\lambda^2+9.\end{aligned} \tag{9}$$

A general formula for even $n=2p$ follows from the Fourier transform of $\cosh^{-(2j+1)}(y)$ [11]:

$$M_{2p}(\lambda)=(-1)^p\frac{(2p)!}{\pi}\sum_{q=0}^{p}\binom{p}{q}\frac{(-1)^q}{(2q)!}\prod_{m=1}^{q}[\lambda^2+(2m+1)^2]. \tag{10}$$

Some other properties of the polynomials $M_n(\lambda)$ are as follows.

(i) The operator $J(\lambda)=\tan(\mathrm{d}/\mathrm{d}\lambda)$ shifts the index according to

$$J(\lambda)M_n(\lambda)=nM_{n-1}(\lambda). \tag{11}$$

In the classification of Sheffer [12] the $M_n(\lambda)$ thus belong to the A-type zero class.

(ii) Partial integration in (8) gives a Rodrigues-type formula:

$$M_n(\lambda)=\frac{n!}{\pi}\cosh(\pi\lambda/2)\left(J^n(\lambda)\frac{\pi}{2\cosh(\pi\lambda/2)}\right). \tag{12}$$

(iii) Combining the recursion relation (4) with (11) one obtains the differential equation:

$$[(n-1)J^2(\lambda)-\lambda J(\lambda)+n]M_n(\lambda)=0. \tag{13}$$

(iv) The orthogonality among polynomials is [10]

$$\int_{-\infty}^{\infty}M_n(\lambda)M_n(\lambda)\frac{\mathrm{d}\lambda}{2\cosh(\pi\lambda/2)}=n!\delta_{nm}. \tag{14}$$

For our problem we are mainly interested in the zeros of the $M_n(\lambda)$ which give the eigenvalues λ_ν for the finite-N case. For each λ_ν, $(-\lambda_\nu)$ is also an eigenvalue. The spectrum differs for even and odd N; in the latter case it contains $\lambda=0$. In the following we confine ourselves to even N. Physically, this corresponds to fixed boundary conditions for one sublattice and free ones for the other sublattice, as shown in figure 1. Then with new fermion operators α_ν and β_ν where

$$\alpha_\nu=\mathcal{N}_\nu\sum_n\frac{1}{n!}M_n(\lambda_\nu)c_n \qquad \text{for } \lambda_\nu>0$$

(where $\mathcal{N}_\nu$ is the appropriate normalisation factor) and β_ν is the corresponding hole operator for $\lambda_\nu<0$, $\mathcal{H}$ can be written as

$$\mathcal{H}=\sum_\nu\lambda_\nu(\alpha_\nu^+\alpha_\nu+\beta_\nu^+\beta_\nu)+\text{constant} \tag{15}$$

so that the ground state of $\mathcal{H}$ is the vacuum of the new fermions α_ν, β_ν. The two (equivalent) sublattices are reflected in the two types of operators. We observe that the canonical transformation to the new fermions is, for $N\to\infty$, a critical limit (i.e. $k\to 1$) of the transformation given by Thacker and Itoyama [13].

A general formula for the λ_ν does not seem to exist. However, one can find an expression for small λ_ν and $N\to\infty$ from (8). In this case the main contribution to the integral comes from the maximum of $\tanh^N(y)/\cosh(y)$ which is located at $y=\frac{1}{2}\ln(4N)$ and where the width of the peak is of order one. For $\ln N\gg 1$, one obtains

$$2\lambda_\nu\simeq\frac{2\pi}{\ln N}(2\nu-1) \qquad \nu=1,2,\ldots. \tag{16}$$

This is exactly the result predicted using conformal arguments for the present anisotropic limit [3]. In numerical calculations [3, 6] the condition $\ln N\gg 1$ normally cannot be reached and one has corrections which change $\ln N$ into $\ln(\beta N)$.

The same approach can, of course, be used for a homogeneous spin chain which is related to the usual row-to-row transfer matrix. Then one encounters the Tchebyscheff polynomials, the zeros of which can be found analytically. The CTM studied in [6] is an intermediate case and corresponds to restricting n in (1) to $M \leqslant n \leqslant N$. The polynomials in this case seem to be more complicated, although it is easier to study the problem numerically. Finally, we mention that the present finite-size approach using orthogonal polynomials can also be applied to Ising systems away from the critical point. One then has to study an anisotropic XY-spin chain with linearly increasing couplings. This will be treated in a separate publication.

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